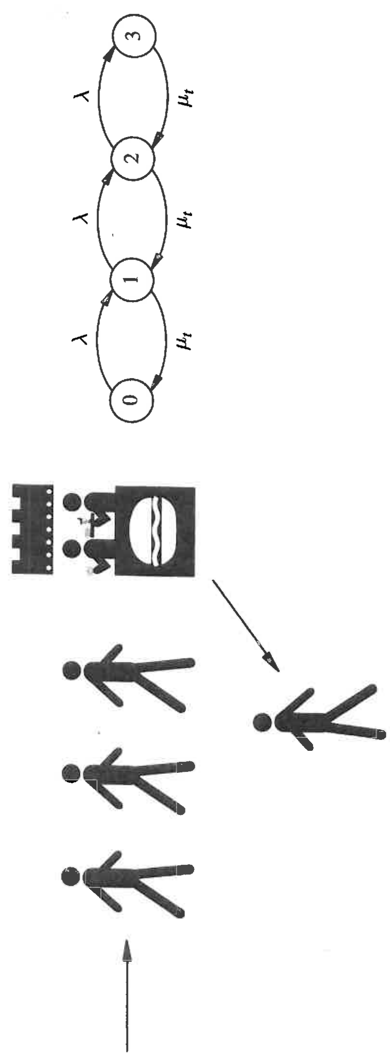


(a) System diagram

(b) State transition diagram

Figure 2.1 Diagrams for system TAF (twice as fast)

Substituting $\lambda = \frac{1}{2}$ and $\mu_t = 1$, we can find the steady-state probabilities of being in any state of the system.

$$P_0 = \frac{8}{15}$$
$$P_1 = \frac{4}{15}$$
$$P_2 = \frac{2}{15}$$
$$P_3 = \frac{1}{15}$$

We are considering this system to be a single server comprised of two workers. The utilization of the server is given by $1 - P_0$, where P_0 is the probability that the server is idle. Therefore, utilization of the server is

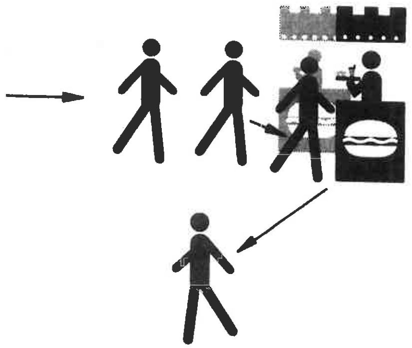
$$\begin{aligned} U_{\text{TAF}} &= 1 - P_0 \\ &= 1 - \frac{8}{15} \\ &= \frac{7}{15} \end{aligned}$$

This means that $\frac{7}{15}$ of the time, the twice-as-fast server has something to do. Now, using the utilization law, $U = XD$, we can find the throughput of system TAF as follows:

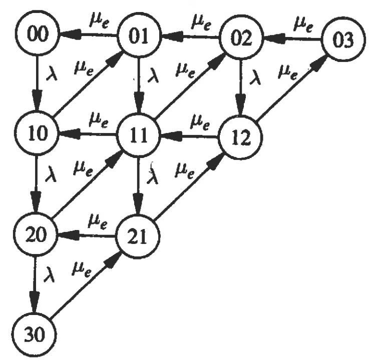
$$\begin{aligned} X_{\text{TAF}} &= \frac{U_{\text{TAF}}}{D_t} \\ &= \frac{7/15}{1} \\ &= \frac{7}{15} \end{aligned}$$

2.2.2 Sequential System

Ms. Droll has suggested the use of two lines, as shown in Figure 2.2(a). Customers will first place their orders at one line, then wait in another line for their orders to be filled. We call this system SEQ since the two lines are visited sequentially by customers. The state diagram for system SEQ is a little more complicated than for system TAF and is shown in Figure 2.2(b). The state of



(a) System diagram



(b) State transition diagram

Figure 2.2 Diagrams for system SEQ (sequential)

the system can no longer be expressed as the length of a single line in the restaurant. Instead, the state depends on the length of both the lines. Let ij indicate a state in which there are i people in the first line placing orders and j people in the second line picking up orders. There are 10 possible states: 00, 01, 02, 03, 10, 11, 12, 20, 21, and 30.

Customers entering the store at rate λ must first be served at the first line, where they place their order and pay at rate μ_e. After finishing at the first line, customers move to the second line, where they pick up their order, and then leave the store at rate μ_e. The system of balance equations that comes from the diagram is

$$\begin{aligned}
\lambda P_{00} &= \mu_e P_{01} \\
(\lambda + \mu_e) P_{01} &= \mu_e P_{02} + \mu_e P_{10} \\
(\lambda + \mu_e) P_{02} &= \mu_e P_{03} + \mu_e P_{11} \\
\mu_e P_{03} &= \mu_e P_{12} \\
(\lambda + \mu_e) P_{10} &= \lambda P_{00} + \mu_e P_{11} \\
(\lambda + 2\mu_e) P_{11} &= \lambda P_{01} + \mu_e P_{12} + \mu_e P_{20} \\
2\mu_e P_{12} &= \lambda P_{02} + \mu_e P_{21} \\
(\lambda + \mu_e) P_{20} &= \lambda P_{10} + \mu_e P_{21} \\
2\mu_e P_{21} &= \lambda P_{11} + \mu_e P_{30} \\
\mu_e P_{30} &= \lambda P_{20} \\
\sum P_{ij} &= 1
\end{aligned}$$

As before, one of the first 10 equations is redundant. The solution to these balance equations is

$$\begin{aligned}
P_{00} &= \frac{\mu_e^3}{4\lambda^3 + 3\lambda^2\mu_e + 2\lambda\mu_e^2 + \mu_e^3} \\
P_{01} = P_{10} &= \frac{\lambda\mu_e^2}{4\lambda^3 + 3\lambda^2\mu_e + 2\lambda\mu_e^2 + \mu_e^3} \\
P_{02} = P_{11} = P_{20} &= \frac{\lambda^2\mu_e}{4\lambda^3 + 3\lambda^2\mu_e + 2\lambda\mu_e^2 + \mu_e^3} \\
P_{03} = P_{12} = P_{21} = P_{30} &= \frac{\lambda^3}{4\lambda^3 + 3\lambda^2\mu_e + 2\lambda\mu_e^2 + \mu_e^3}
\end{aligned}$$

We notice that many states have the same solution: $P_{10} = P_{01}$, $P_{20} = P_{11} = P_{02}$ and $P_{30} = P_{21} = P_{12} = P_{03}$. Identifying these sets of equivalent states in the diagram, we discover that for each set, the states fall on the same diagonal line. This kind of symmetry is typical. Failure to find the symmetry in a solution is often good reason to recheck one's work. This is not to say, however, that all systems display such symmetry.

Finally, substituting values for $\lambda = \frac{1}{2}$ and $\mu_e = 1$, the steady-state probabilities for this particular system are

$$\begin{aligned}
P_{00} &= \frac{4}{13} \\
P_{10} = P_{01} &= \frac{2}{13} \\
P_{20} = P_{11} = P_{02} &= \frac{1}{13} \\
P_{30} = P_{21} = P_{12} = P_{03} &= \frac{1}{26}
\end{aligned}$$

Again, we use the utilization law, $U = XD$, to determine the throughput of this system. There are two servers in this system, though. Do we use the utilization and demand of the first server, the second server, or somehow "average" them together?

The answer comes by looking at the *topology* of the system, meaning the connections between the servers. The topology of system SEQ is a straight line: customers (A) enter the restaurant, (B) are served by the first server (place order), (C) are served by the second server (pick up order), and (D) exit the store. This sequence of events is shown in Figure 2.3. Remember that we have assumed that our system is in steady-state. A consequence of this assumption is that, for any state, flow into that state is equal to flow out of that state. A generalization of this concept is that flow into the system must be equal to flow out of the system. Since the topology of this system is strictly sequential, we

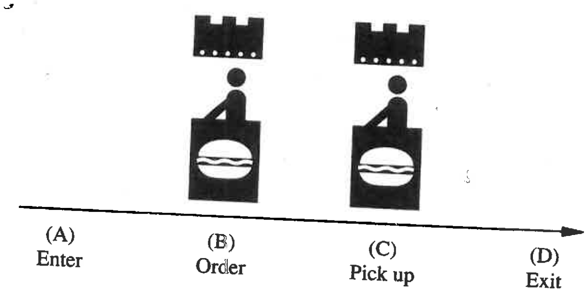


Figure 2.3 Topology of system SEQ

can measure throughput at any point along the line shown in the figure and it will be the same. We know the utilizations and demands for points B and C, so either of these points will do. Let us consider point B, which is the first server.

Utilization of the first server, U_{SEQ1}, can be obtained by summing the probabilities of all states in which there are no customers at the first server (idle time), and subtracting this sum from 1. Any state that has a zero as the first digit of its label is included in the summation.

$$\begin{aligned} U_{\text{SEQ1}} &= 1 - (P_{00} + P_{01} + P_{02} + P_{03}) \\ &= 1 - \frac{4}{13} - \frac{2}{13} - \frac{1}{13} - \frac{1}{26} \\ &= \frac{11}{26} \end{aligned}$$

Demand at the first server is D_e, which is 1. Therefore, throughput of the first server is given by

$$\begin{aligned} X_{\text{SEQ1}} &= \frac{U_{\text{SEQ1}}}{D_e} \\ &= \frac{11}{26} \end{aligned}$$

which is the throughput of the system. Let us compute throughput at point C to verify that it will be the same. Utilization of the second server, U_{SEQ2}, is obtained by summing the probabilities of all states having a zero as the second digit in the state label and subtracting this sum from 1.

$$\begin{aligned} U_{\text{SEQ2}} &= 1 - (P_{00} + P_{10} + P_{20} + P_{30}) \\ &= 1 - \frac{4}{13} - \frac{2}{13} - \frac{1}{13} - \frac{1}{26} \\ &= \frac{11}{26} \end{aligned}$$

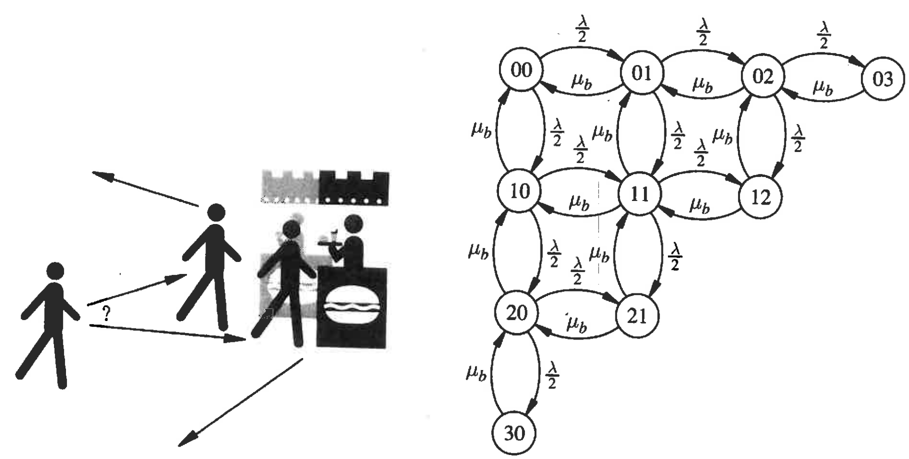
Demand at the second server is also D_e, which is 1. Therefore, throughput of the second server is

$$\begin{aligned} X_{\text{SEQ2}} &= \frac{U_{\text{SEQ2}}}{D_e} \\ &= \frac{11}{26} \end{aligned}$$

which is the same as X_{SEQ1}.

2.2.3 Random-Line System

Mr. Innis described a system in which there are two full-service lines, each served by a single worker, as shown in Figure 2.4(a). Customers enter the store and randomly choose a line; therefore, we will refer to this system as system RL. We shall interpret the word *random* to mean that the probability of choosing a line is the same for all lines. Since there are two lines in this system, then the probability of choosing any particular line is $\frac{1}{2}$. Therefore, customers entering the store arrive at either line with rate $\lambda/2$. Each line is full-service in the sense that customers place their orders and pick them up in the same line at rate μ_b.



(a) System diagram (b) State transition diagram

Figure 2.4 Diagrams for system RL (random line)

The state diagram for system RL is shown in Figure 2.4(b). Notice that the system has the same set of states as system SEQ, but the transitions between states are quite different. The system of balance equations for system RL is

$$
\begin{aligned}
\lambda P_{00} &= \mu_b P_{01} + \mu_b P_{10} \\
(\lambda + \mu_b) P_{01} &= \frac{\lambda}{2} P_{00} + \mu_b P_{02} + \mu_b P_{11} \\
(\lambda + \mu_b) P_{02} &= \frac{\lambda}{2} P_{01} + \mu_b P_{03} + \mu_b P_{12} \\
\mu_b P_{03} &= \frac{\lambda}{2} P_{02} \\
(\lambda + \mu_b) P_{10} &= \frac{\lambda}{2} P_{00} + \mu_b P_{11} + \mu_b P_{20} \\
(\lambda + 2\mu_b) P_{11} &= \frac{\lambda}{2} P_{01} + \frac{\lambda}{2} P_{10} + \mu_b P_{12} + \mu_b P_{21} \\
2\mu_b P_{12} &= \frac{\lambda}{2} P_{02} + \frac{\lambda}{2} P_{11} \\
(\lambda + \mu_b) P_{20} &= \frac{\lambda}{2} P_{10} + \mu_b P_{30} + \mu_b P_{21} \\
2\mu_b P_{21} &= \frac{\lambda}{2} P_{20} + \frac{\lambda}{2} P_{11} \\
\mu_b P_{30} &= \frac{\lambda}{2} P_{20} \\
\sum P_{ij} &= 1
\end{aligned}
$$

> All these equations "look" bad but are not difficult. First, study the state diagram and verify its correctness. Then these equations can be written directly from the diagram.

The steady-state solution for system RL is

$$
\begin{aligned}
P_{00} &= \frac{4\mu_b^3}{2\lambda^3 + 3\lambda^2\mu_b + 4\lambda\mu_b^2 + 4\mu_b^3} \\
P_{01} = P_{10} &= \frac{2\lambda\mu_b^2}{2\lambda^3 + 3\lambda^2\mu_b + 4\lambda\mu_b^2 + 4\mu_b^3} \\
P_{02} = P_{11} = P_{20} &= \frac{\lambda^2\mu_b}{2\lambda^3 + 3\lambda^2\mu_b + 4\lambda\mu_b^2 + 4\mu_b^3} \\
P_{03} = P_{12} = P_{21} = P_{30} &= \frac{\frac{\lambda^3}{2}}{2\lambda^3 + 3\lambda^2\mu_b + 4\lambda\mu_b^2 + 4\mu_b^3}
\end{aligned}
$$

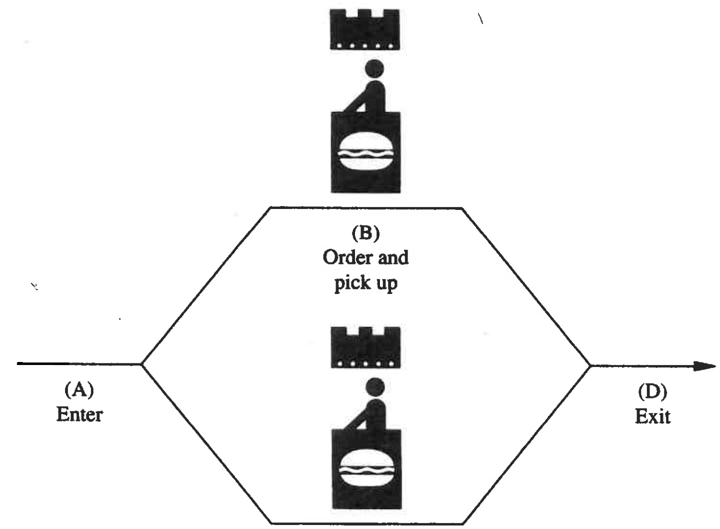


Figure 2.5 Topology of system RL

Substituting $\lambda = \frac{1}{2}$ and $\mu_b = \frac{1}{2}$ the steady-state probabilities for system RL are

$$
\begin{aligned}
P_{00} &= \frac{4}{13} \\
P_{01} = P_{10} &= \frac{2}{13} \\
P_{02} = P_{11} = P_{20} &= \frac{1}{13} \\
P_{03} = P_{12} = P_{21} = P_{30} &= \frac{1}{26}
\end{aligned}
$$

This is the same solution as for system SEQ, but are the throughputs of the two systems also the same?

To find the throughput of system RL we must again consider the system topology, which is shown in Figure 2.5. Every customer will take either path (A,B,D) or (A,C,D) through the system. Therefore, to get the throughput of the system we must measure throughput at both points B and C, the two servers, and add them together.

$$
\begin{aligned}
U_{RL1} &= 1 - (P_{00} + P_{01} + P_{02} + P_{03}) \\
&= 1 - \frac{4}{13} - \frac{2}{13} - \frac{1}{13} - \frac{1}{26}
\end{aligned}
$$

$$= \frac{11}{26}$$

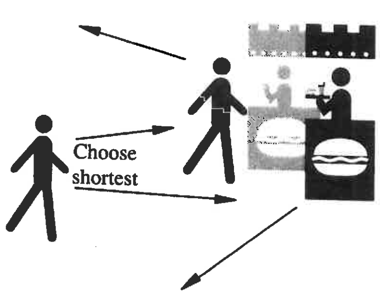
$$\begin{aligned} U_{RL2} &= 1 - (P_{00} + P_{10} + P_{20} + P_{30}) \\ &= 1 - \frac{4}{13} - \frac{2}{13} - \frac{1}{13} - \frac{1}{26} \\ &= \frac{11}{26} \end{aligned}$$

$$\begin{aligned} X_{RL} &= X_{RL1} + X_{RL2} \\ &= \frac{U_{RL1}}{D_b} + \frac{U_{RL2}}{D_b} \\ &= \frac{11/26}{2} + \frac{11/26}{2} \\ &= \frac{11}{26} \end{aligned}$$

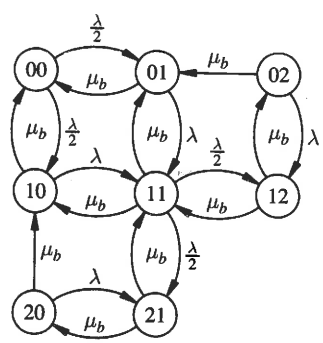
We see that the throughputs of system SEQ and system RL are the same.

2.2.4 Shortest Line System

Mrs. Gardener pointed out to Mr. Innis and the rest of the board that customers are more likely to choose the shortest line to get into when they enter the restaurant. Her system is exactly like Mr. Innis's except that when lengths of the two lines are different, an arriving customer will get into the shorter line, as shown in Figure 2.6(a). When the line lengths are the same, there is an equal probability of a customer getting into any particular line. The state diagram for



(a) System diagram



(b) State transition diagram

Figure 2.6 Diagrams for system SL (shortest line)

this system, which we will refer to as system SL, is given in Figure 2.6(b). The system of balance equations is

$$\begin{aligned} \lambda P_{00} &= \mu_b P_{01} + \mu_b P_{10} \\ (\lambda + \mu_b) P_{01} &= \frac{\lambda}{2} P_{00} + \mu_b P_{02} + \mu_b P_{11} \\ (\lambda + \mu_b) P_{02} &= \mu_b P_{12} \\ (\lambda + \mu_b) P_{10} &= \frac{\lambda}{2} P_{00} + \mu_b P_{20} + \mu_b P_{11} \\ (\lambda + 2\mu_b) P_{11} &= \lambda P_{01} + \lambda P_{10} + \mu_b P_{12} + \mu_b P_{21} \\ 2\mu_b P_{12} &= \lambda P_{02} + \frac{\lambda}{2} P_{11} \\ (\lambda + \mu_b) P_{20} &= \mu_b P_{21} \\ 2\mu_b P_{21} &= \frac{\lambda}{2} P_{11} + \lambda P_{20} \\ \sum P_{ij} &= 1 \end{aligned}$$

The solution set is

$$\begin{aligned} P_{00} &= \frac{\mu_b^3(3\lambda + 4\mu_b)}{\lambda^4 + 3\lambda^3\mu_b + 5\lambda^2\mu_b^2 + 7\lambda\mu_b^3 + 4\mu_b^4} \\ P_{01} = P_{10} &= \frac{[\lambda\mu_b^2(3\lambda + 4\mu_b)]/2}{\lambda^4 + 3\lambda^3\mu_b + 5\lambda^2\mu_b^2 + 7\lambda\mu_b^3 + 4\mu_b^4} \\ P_{02} = P_{20} &= \frac{\lambda^3\mu_b/2}{\lambda^4 + 3\lambda^3\mu_b + 5\lambda^2\mu_b^2 + 7\lambda\mu_b^3 + 4\mu_b^4} \\ P_{11} &= \frac{\lambda^2\mu_b(\lambda + 2\mu_b)}{\lambda^4 + 3\lambda^3\mu_b + 5\lambda^2\mu_b^2 + 7\lambda\mu_b^3 + 4\mu_b^4} \\ P_{12} = P_{21} &= \frac{[\lambda^3(\lambda + \mu_b)]/2}{\lambda^4 + 3\lambda^3\mu_b + 5\lambda^2\mu_b^2 + 7\lambda\mu_b^3 + 4\mu_b^4} \end{aligned}$$

With $\lambda = \frac{1}{2}$ and $\mu_b = \frac{1}{2}$, the steady-state probabilities for system SL are:

$$\begin{aligned} P_{00} &= \frac{7}{20} \\ P_{01} = P_{10} &= \frac{7}{40} \\ P_{02} = P_{20} &= \frac{1}{40} \end{aligned}$$

$$P_{11} = \frac{3}{20}$$

$$P_{12} = P_{21} = \frac{1}{20}$$

Since the topology of system SL is the same as system RL (see Figure 2.5), throughput is found in the same way. That is, we find the throughputs of the two servers and add them together.

$$\begin{aligned} U_{SL1} &= 1 - (P_{00} + P_{01} + P_{02}) \\ &= 1 - \frac{7}{20} - \frac{7}{40} - \frac{1}{40} \\ &= \frac{9}{20} \end{aligned}$$

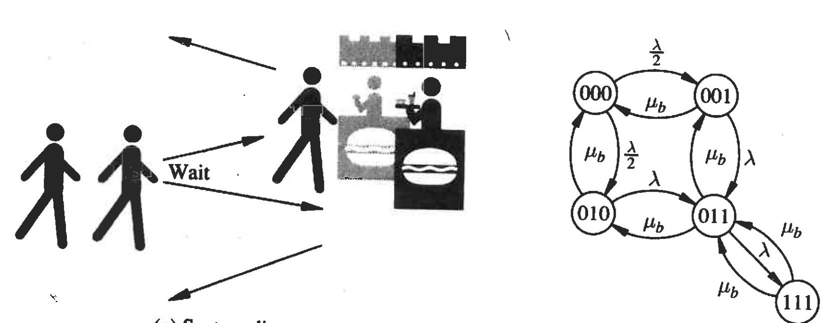
$$\begin{aligned} U_{SL2} &= 1 - (P_{00} + P_{10} + P_{20}) \\ &= 1 - \frac{7}{20} - \frac{7}{40} - \frac{1}{40} \\ &= \frac{9}{20} \end{aligned}$$

$$\begin{aligned} X_{SL} &= X_{SL1} + X_{SL2} \\ &= \frac{U_{SL1}}{D_b} + \frac{U_{SL2}}{D_b} \\ &= \frac{9/20}{2} + \frac{9/20}{2} \\ &= \frac{9}{20} \end{aligned}$$

2.2.5 Common-Line System

Freddy, who often sits in on board meetings, suggested a system that is popular in banks and post offices. In this system, shown in Figure 2.7(a), there is one long line where everyone waits. When a worker becomes free, the person at the head of the line goes to that worker to be served. All the workers draw customers from a common, shared line, which is why Freddy calls this the common-line system. We will refer to it as system CL.

Figure 2.7(b) shows the state diagram for this system. The state description in this diagram is slightly different from the previous three. Each state is marked



(a) System diagram (b) State transition diagram

Figure 2.7 Diagrams for system CL (common line)

with three numbers, for example, ijk, where i is the number of people waiting in the common line, j is 1 or 0 depending on whether or not the first worker is busy serving a customer, and k (like j) is 1 or 0 depending on whether the second worker is busy serving a customer. The balance equations for system CL are

$$\begin{aligned} \lambda P_{000} &= \mu_b P_{001} + \mu_b P_{010} \\ (\lambda + \mu_b) P_{001} &= \frac{\lambda}{2} P_{000} + \mu_b P_{011} \\ (\lambda + \mu_b) P_{010} &= \frac{\lambda}{2} P_{000} + \mu_b P_{011} \\ (\lambda + 2\mu_b) P_{011} &= \lambda P_{001} + \lambda P_{010} + 2\mu_b P_{111} \\ 2\mu_b P_{111} &= \lambda P_{011} \\ \sum P_{ijk} &= 1 \end{aligned}$$

The steady-state solution is

$$\begin{aligned} P_{000} &= \frac{4\mu_b^3}{\lambda^3 + 2\lambda^2\mu_b + 4\lambda\mu_b^2 + 4\mu_b^3} \\ P_{001} = P_{010} &= \frac{2\lambda\mu_b^2}{\lambda^3 + 2\lambda^2\mu_b + 4\lambda\mu_b^2 + 4\mu_b^3} \\ P_{011} &= \frac{2\lambda^2\mu_b}{\lambda^3 + 2\lambda^2\mu_b + 4\lambda\mu_b^2 + 4\mu_b^3} \\ P_{111} &= \frac{\lambda^3}{\lambda^3 + 2\lambda^2\mu_b + 4\lambda\mu_b^2 + 4\mu_b^3} \end{aligned}$$

With $\lambda = \frac{1}{2}$ and $\mu_b = \frac{1}{2}$, the steady state probabilities for system CL are:

$$P_{000} = \frac{4}{11}$$
$$P_{001} = P_{010} = P_{011} = \frac{2}{11}$$
$$P_{111} = \frac{1}{11}$$

Throughput of system CL is

$$\begin{aligned} U_{CL1} &= 1 - (P_{000} + P_{001}) \\ &= 1 - \frac{4}{11} - \frac{2}{11} \\ &= \frac{5}{11} \end{aligned}$$

$$\begin{aligned} U_{CL2} &= 1 - (P_{000} + P_{010}) \\ &= 1 - \frac{4}{11} - \frac{2}{11} \\ &= \frac{5}{11} \end{aligned}$$

$$\begin{aligned} X_{CL} &= X_{CL1} + X_{CL2} \\ &= \frac{U_{CL1}}{D_b} + \frac{U_{CL2}}{D_b} \\ &= \frac{5/11}{2} + \frac{5/11}{2} \\ &= \frac{5}{11} \end{aligned}$$

2.3 COMPARING THE ALTERNATIVES

Now that we have solved for the steady-state probabilities for each of the proposed alternatives, we can compare them to see which is best. But what do we mean by *best*? That is, what is the most appropriate objective function to use in determining a ranking of systems with respect to desirability? In the preceding problem, the best worker was hired depending on the throughput of the sno-cone

stand when he or she worked. We can apply a similar notion of *best* to the JiffyBurger problem. That is, the system that has the highest throughput is the best. The highest throughput will mean the most customers being served per minute, hence more money made.

But there are other measures of *best* that should also be considered. Suppose that a system has high throughput, but customers spend a lot of time in the restaurant. Over time, customers may not like coming to JiffyBurger because it takes too long to get in and out. Thus, as a function of time, λ may decrease due to dissatisfied customers. The length of time between entering the restaurant and leaving it is known as the *response time* of the system. It is the average amount of time a customer spends in the restaurant. Hence, another objective function would be to minimize the response time. This will undoubtedly please the customers, which may result in an increased arrival rate λ.

Another way to rank systems would be to compare *wait times*, that is, the amount of time a customer must wait while not receiving service. Customers do not seem to mind standing in line as long as they know they are being waited on. Wait time is simply the response time of the system less the average service time.

So now we have three different objective functions by which to rank these five systems. Instead of deciding which of these is the proper "best" criterion, we will find all three rankings, present them to the board, and let them decide which is most appropriate.

> Other objective functions are feasible. Examples include minimizing server idle time, minimizing the number of customers who go to NiftyBurger, minimizing the variance between customer response times, and so on. The proper objective function is always a management decision.

2.3.1 Ranking by Throughput

Table 2.1 shows the throughputs of each system. We see that system TAF is the best (having the highest throughput), followed by system CL, system SL, system RL, and system SEQ. Intuitively, this makes sense. The best throughput is achieved when all of the server "power" is concentrated in one place. Whenever there is anyone in the restaurant, the server (both workers working together) is utilized. The other schemes, however, allow for a worker to be idle while the restaurant is not empty.

System CL has better throughput than system SL, but is the difference really significant? A look at the percentage improvement of one system over another shows that system CL delivers only 0.8% more throughput than system SL. Comparing system CL to system SEQ shows an improvement of 7.3%.